Heisenberg Spins on an Elastic Torus Section*

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Classical Heisenberg spins in the continuum limit (i.e. the nonlinear σ-model) are studied on an elastic torus section with homogeneous boundary conditions. The corresponding rigid model exhibits topological soliton configurations with geometrical frustration due to the torus eccentricity. Assuming small and smooth deformations allows to find shapes of the elastic support by relaxing the rigidity constraint: an inhomogeneous Lamé equation arises. Finally, this leads to a novel geometric effect: a *global shrinking* with swellings.
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Curved magnetic structures abound in nature: either made out of magnetic materials or enclosing magnetorheological fluids. To explore their magnetoelastic properties, it is convenient to treat their surfaces as a continuum of classical spins. Keeping this in mind, we investigate classical Heisenberg-coupled spins on a deformable torus section in the presence of topological spin solitons.

The continuum limit of the Heisenberg Hamiltonian for classical ferromagnets or antiferromagnets for isotropic spin-spin coupling is the nonlinear σ-model [1–5]. The total Hamiltonian for a deformable, magnetoelastically coupled manifold is given by $\mathcal{H} = \mathcal{H}_{magn} + \mathcal{H}_{el} + \mathcal{H}_{m-el}$, where $\mathcal{H}_{magn}$, $\mathcal{H}_{el}$ and $\mathcal{H}_{m-el}$ represent the magnetic, elastic and magnetoelastic energy, respectively. In the present paper we will focus on the magnetic part and the elastic part only. For the nonlinear σ-model, the magnetic energy on a curved surface $\mathcal{S}$, in curvilinear coordinates, is given by [6,7]

$$\mathcal{H}_{magn} = J \iint_{\mathcal{S}} \sqrt{g}\, \mathrm{d}\Omega\, g^{ij} h_{\alpha\beta} \partial_i n^\alpha \partial_j n^\beta, \tag{1}$$

where J denotes the coupling energy between neighboring spins. The order parameter $\widehat{\mathbf{n}}$ is the local magnetization unit vector specified by a point on the sphere S^2. The metric tensors (g_{ij}) and $(h_{\alpha\beta})$ describe respectively the support surface $\mathcal{S}$ and the order parameter manifold: as customary, $\mathrm{d}\Omega$ represents the surface area and g the determinant $\det(g_{ij})$.

First, let us consider the nonlinear σ-model on a rigid torus section. The most natural representation of a torus is given in cylindrical coordinates (ρ, ξ, z):

$$\rho = R + r\cos\varphi, \quad z = r\sin\varphi, \tag{2}$$

where the *rotating radius* R and the *axial radius* r must verify $0 < r < R$, while the angle φ varies from $-\pi$ to π. Nevertheless, for our purposes we will use a more suitable representation [8]

$$\rho = \frac{a\sinh b}{\cosh b - \cos\eta}, \quad z = \frac{a\sin\eta}{\cosh b - \cos\eta}, \tag{3}$$

where the new constant parameters a and b are both real and positive, while the new angle η varies from $-\pi$ to π. The relations

$$a = \sqrt{(R+r)(R-r)} \quad \text{and} \quad \cosh b = \frac{R}{r} \tag{4}$$

allow a simple geometrical interpretation for the new parameters: let us call a the *geometric radius* and b the *eccentric angle*. Conversely, the natural parameters R and r satisfy

$$R = \frac{a}{\tanh b} \quad \text{and} \quad r = \frac{a}{\sinh b}. \tag{5}$$

Furthermore, the transformation yields

$$\tan\tfrac{1}{2}\eta = \tanh\tfrac{1}{2}b\ \tan\tfrac{1}{2}\varphi, \tag{6a}$$

$$= \sqrt{\frac{R-r}{R+r}}\tan\tfrac{1}{2}\varphi. \tag{6b}$$

One can easily check that the metric on a torus in *peripolar coordinates* (ξ, η) is given by

$$g = \frac{a^2}{(\cosh b - \cos\eta)^2}\left[\sinh^2 b\ \mathrm{d}\xi\otimes\mathrm{d}\xi + \mathrm{d}\eta\otimes\mathrm{d}\eta\right], \tag{7}$$

therefore $g^{\xi\eta} = g^{\eta\xi} = 0$ and we have

$$g^{\xi\xi}\sqrt{g} = \frac{1}{\sinh b}, \quad g^{\eta\eta}\sqrt{g} = \sinh b. \tag{8}$$

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From now on, we restrict ourselves only to a section of the torus: the angle ξ about the z-axis will vary from $-\Delta\xi$ to $\Delta\xi$ where $0 < \Delta\xi < \pi$.

As usual, the local magnetization $\widehat{\mathbf{n}}$ is described by its polar coordinates (Θ, Φ), then the metric on the Heisenberg sphere is given by

$$h = \mathrm{d}\Theta \otimes \mathrm{d}\Theta + \sin^2\Theta \; \mathrm{d}\Phi \otimes \mathrm{d}\Phi. \tag{9}$$

Assuming homogeneous boundary conditions at both ends ($\Theta = 0\,[\pi]$ as $\xi \to \pm\Delta\xi$) allows to map each boundary of the section to a point: thus we compactify our torus section into the sphere S^2. Consequently, the mapping of our support to the order parameter manifold is classified by the homotopy group $\Pi_2\left(S^2\right)$ which is isomorphic to $\mathbb{Z}$: spin configurations may be classified according to their homotopy class [1,9].

Henceforth, without loss of generality, only toroidal symmetric configurations ($\partial_\eta\Theta = \partial_\xi\Phi = 0$) will be considered. Thus the magnetic Hamiltonian (1) becomes

$$\mathcal{H}_{magn} = J \int_{-\Delta\xi}^{+\Delta\xi} \mathrm{d}\xi \int_{-\pi}^{+\pi} \mathrm{d}\eta \left[\frac{\Theta_\xi^2}{\sinh b} + \sinh b \sin^2\Theta \; \Phi_\eta^2 \right], \tag{10}$$

where a subscript stands for differentiation. Rescaling the angle about z-axis in equation (10) gives:

$$\mathcal{H}_{magn} = J \int_{-\Delta\zeta}^{+\Delta\zeta} \mathrm{d}\zeta \int_{-\pi}^{+\pi} \mathrm{d}\eta \left[\Theta_\zeta^2 + \sin^2\Theta \; \Phi_\eta^2 \right], \tag{11}$$

where $\zeta \equiv \sinh b \; \xi$ and $\Delta\zeta \equiv \sinh b \; \Delta\xi$. The Euler-Lagrange equations corresponding to (11) are:

$$\Phi_{\eta\eta} = 0, \tag{12a}$$
$$\Theta_{\zeta\zeta} = \Phi_\eta^2 \sin\Theta\cos\Theta. \tag{12b}$$

From (12a), it follows that

$$\Phi_\eta = q_\eta \qquad q_\eta \in \mathbb{Z}. \tag{13a}$$

Substituting this into (12b) and rescaling again the rotating angle, we get the sine-Gordon (SG) equation

$$\Theta_{\varrho\varrho} = \sin\Theta\cos\Theta, \tag{13b}$$

where $\varrho \equiv q_\eta\zeta$, or $\varrho \equiv q_\eta \sinh b \; \xi$.

Equation (13b) may be integrated once to yield

$$\Theta_\varrho^2 = \sin^2\Theta + \widetilde{m} \qquad \widetilde{m} \in [0, +\infty). \tag{14}$$

The limit case $\widetilde{m} = 0$ corresponds to the self-duality equation [7]. Performing the change of variable $\sin\Theta = \mathrm{dn}(u \mid 1+\widetilde{m})$ where dn is a Jacobi elliptic function [10], the differential equation becomes $u_\varrho^2 = 1$. Let us denote by $\alpha(\cdot \mid \widetilde{m})$ the increasing solution of (13b) specified by the *parameter* $\widetilde{m}$ and subject to the boundary condition $\alpha(0 \mid \widetilde{m}) = \frac{\pi}{2}$. Readily, we get:

$$\sin\alpha(\varrho \mid \widetilde{m}) = \mathrm{dn}(\varrho \mid 1+\widetilde{m}). \tag{15}$$

Simple calculations lead to the cosine variant:

$$\cos\alpha(\varrho \mid \widetilde{m}) = -\sqrt{1+\widetilde{m}} \; \mathrm{sn}(\varrho \mid 1+\widetilde{m}), \tag{16a}$$
$$= -\,\mathrm{sn}\Big(\varrho\sqrt{1+\widetilde{m}} \mid 1/1+\widetilde{m}\Big). \tag{16b}$$

Thus, the general SG solution, in natural coordinates, is

$$\theta(\varrho) = \varepsilon\alpha(\varrho \mid \widetilde{m}) + \mathsf{c} \qquad \varepsilon = \pm 1. \tag{17}$$

Further the function $\alpha(\cdot \mid \widetilde{m})$ satisfies

$$\alpha(\varrho + 2qK_\alpha \mid \widetilde{m}) = \alpha(\varrho \mid \widetilde{m}) + q\pi \qquad q \in \mathbb{Z}, \tag{18}$$

where the *quasi quarter-period* K_α is related to the complete elliptic integral of the first kind K by [10]

$$K_\alpha(\widetilde{m}) = \mathrm{K}(1+\widetilde{m}), \tag{19a}$$
$$= \frac{1}{\sqrt{1+\widetilde{m}}} \; \mathrm{K}\Big(\frac{1}{1+\widetilde{m}}\Big). \tag{19b}$$

Using the solutions of equation (13b), the $q_\xi\pi$-soliton configuration consistent with the boundary conditions can be obtained easily. Up to an irrelevant additive multiple of π, we have

$$\Theta(\xi) = \varepsilon\alpha(q_\eta \sinh b \; \xi \mid \widetilde{m}) - \delta_{\mathrm{even},q_\xi} \tfrac{\pi}{2}, \tag{20a}$$

where the parameter $\widetilde{m}$ is given by

$$\widetilde{m} = K_\alpha^{-1}\Big(\frac{q_\eta}{q_\xi}\Delta\xi \; \sinh b\Big). \tag{20b}$$

The magnetic energy E_{magn} of the above configuration (20) may be compared with the corresponding topological minimum energy $\underline{\mathrm{E}}$ which does not depend on the geometry of the support manifold. Performing the Bogomol'nyi's decomposition [9] yields

$$\underline{\mathrm{E}} = 8\pi J|Q|, \tag{21}$$

where the topological charge (i.e. the winding number) Q equals to $q_\xi q_\eta$. Let $\mathcal{E}_{magn}$ denote the ratio $\mathrm{E}_{magn}/\underline{\mathrm{E}}$. A straightforward calculation shows that $\mathcal{E}_{magn}$ depends on the parameter $\widetilde{m}$ only; we have

$$\mathcal{E}_{magn} = \mathrm{E}(1+\widetilde{m}) + \tfrac{1}{2}\widetilde{m} \; \mathrm{K}(1+\widetilde{m}), \tag{22a}$$
$$= \sqrt{1+\widetilde{m}} \left[\mathrm{E}\Big(\frac{1}{1+\widetilde{m}}\Big) - \tfrac{1}{2}\frac{\widetilde{m}}{1+\widetilde{m}} \; \mathrm{K}\Big(\frac{1}{1+\widetilde{m}}\Big) \right], \tag{22b}$$

which increases strictly from 1 to ∞ with respect to $\widetilde{m}$; E is the complete elliptic integral of the second kind [10]. To avoid any unnecessary complication, we will consider the quantity $q_\eta\Delta\xi/q_\xi$ fixed. Therewith, according to (20b), $\mathcal{E}_{magn}$ is a decreasing function of the eccentric angle b.

Clearly, the minimum energy $\underline{\mathrm{E}}$ is only reached when b tends to infinity, i.e., when our support becomes an infinite rigid cylinder. This is in agreement with results obtained for the rigid cylinder [6,7]. In other words, the non-satisfaction of the Bogomol'nyi's inequality is due to the geometry of the support manifold, hence the expression: geometrical frustration. Therefore, here, the geometric frustration is induced by introducing a second non-vanishing local curvature. Furthermore, notice that only the eccentric angle, which measures the balance between the rotating radius R and the axial radius r, does tune the spread of the soliton.

Next, let us relax the rigidity constraint and consider the nontrivial spin configuration on an elastic torus section. Accordingly, the soliton will try to minimize its magnetic energy E_{magn} to the minimum energy $\underline{\mathrm{E}}$ by deforming the elastic support [6,7,11].

Since the eccentric angle b appears as the relevant geometric parameter and the rotating angle ξ as the relevant curvilinear coordinate, we relax b with respect to ξ and write

$$b(\xi) = b_0 + \Lambda(\xi)\,, \tag{23}$$

where b_0 represents the *spontaneous eccentric angle* and the function Λ describes local deformations. The metric on the deformable manifold in this case remains orthogonal, and (8) reads

$$g^{\xi\xi}\sqrt{g}=\frac{1}{\sqrt{\sinh^2 b+\Lambda_\xi^2}}, \quad g^{\eta\eta}\sqrt{g}=\sqrt{\sinh^2 b+\Lambda_\xi^2}. \tag{24}$$

As the problem is quasi-one-dimensional, the magnetoelastic energy $\mathcal{H}_{m-el}$ merely renormalizes the spin coupling energy J in the magnetic energy $\mathcal{H}_{magn}$ [12]. Therefore, we add to the nonlinear σ-model Hamiltonian (1) only the elastic energy which is essentially stored in the bending of the deformable support [13–16]:

$$\mathcal{H}_{el} = \tfrac{1}{2}k_c \iint_{\mathcal{S}} \sqrt{g}\,\mathrm{d}\Omega\ \left(H - H_0\right)^2. \tag{25}$$

Here the constant k_c denotes the *bending rigidity*, H represents the mean curvature [17], and the *spontaneous mean curvature* H_0 tends to bias the mean curvature for recovering the spontaneous shape. Assuming small and smooth deformations and expanding to second order in Λ, Λ_ξ and $\Lambda_{\xi\xi}$ lead to [18]

$$\mathcal{H}_{el} = \tfrac{1}{2}\pi k_c c_2 \int_{-\Delta\zeta}^{+\Delta\zeta} \mathrm{d}\zeta\,\Lambda^2, \tag{26a}$$

where

$$c_2 = \frac{1+\sinh b_0 \cosh b_0 \left(3+\sinh^2 b_0\right)}{\sinh^4 b_0}. \tag{26b}$$

Before deriving the Euler-Lagrange equation for the total Hamiltonian $\mathcal{H} = \mathcal{H}_{magn} + \mathcal{H}_{el}$, we calculate the magnetic energy associated with the nontrivial spin configuration (20). Expanding to second order in Λ and Λ_ξ the relations (24) enables to rewrite (11) as follows

$$\begin{aligned}\mathcal{H}_{magn} = J \int_{-\Delta\zeta}^{+\Delta\zeta}\mathrm{d}\zeta\int_{-\pi}^{+\pi}\mathrm{d}\eta\ \Bigg[&\left(1+\tfrac{1}{2}\Lambda^2\right)\left[\Theta_\zeta^2+\sin^2\Theta\ \Phi_\eta^2\right] \\ &-\left(\coth b_0\,\Lambda+\tfrac{1}{2}\Lambda_\zeta^2\right)\left[\Theta_\zeta^2-\sin^2\Theta\ \Phi_\eta^2\right] \\ &+\frac{\Lambda^2\Theta_\zeta^2}{\sinh^2 b_0}\Bigg].\end{aligned} \tag{27}$$

On the other hand, according to (14) and (15), the spin configuration (20) verifies

$$\begin{aligned}&\Theta_\zeta^2 - \sin^2\Theta\ \Phi_\eta^2 = \widetilde{m} q_\eta^2, \\ &\Theta_\zeta^2 = \left(1+\widetilde{m}\right) q_\eta^2 \left[1-\mathrm{sn}^2(q_\eta\zeta \mid 1+\widetilde{m})\right].\end{aligned}$$

Inserting these relations into (27) and simplifying, we obtain

$$\begin{aligned}\mathcal{H}_{magn} = 2\pi J q_\eta^2 \int_{-\Delta\zeta}^{+\Delta\zeta}\mathrm{d}\zeta\ \Bigg[&(2+\widetilde{m}) - \widetilde{m}\left[\coth b_0\,\Lambda+\tfrac{1}{2}\Lambda_\zeta^2\right] \\ &-(1+\widetilde{m})\left[2+\coth^2 b_0\,\Lambda^2\right]\mathrm{sn}^2(q_\eta\zeta \mid 1+\widetilde{m}) \\ &+\tfrac{1}{2}\left(2+\widetilde{m}+2\frac{1+\widetilde{m}}{\sinh^2 b_0}\right)\Lambda^2\Bigg].\end{aligned} \tag{28}$$

The Euler-Lagrange equation for the problem takes the following form

$$\Lambda_{\varrho\varrho} + \left[(1+m)A - mB\ \mathrm{sn}^2(\varrho \mid m)\right]\Lambda = \sqrt{m}\ j, \tag{29a}$$

where we have set

$$m = 1+\widetilde{m}, \tag{29b}$$

$$A = \frac{1}{\widetilde{m} q_\eta^2}\left[1+\frac{2}{2+\widetilde{m}}\left(\frac{1+\widetilde{m}}{\sinh^2 b_0}+\frac{c_2}{\kappa q_\eta^2}\right)\right], \tag{29c}$$

$$B = 2\frac{\coth^2 b_0}{\widetilde{m} q_\eta^2}, \tag{29d}$$

$$j = \frac{\coth b_0}{q_\eta^2\sqrt{1+\widetilde{m}}}. \tag{29e}$$

Here we have introduced the *relative coupling energy* $\kappa \equiv J/k_c$ and the variable $\varrho \equiv q_\eta\zeta$ (see (13b)). The linear inhomogeneous second-order differential equation (29a) is related to the well-known homogeneous (Jacobian) Lamé's equation which occurs in several physical contexts [18,19]. We have found no direct treatment of (29a) in the literature. However, an approach based on the derivation of the Lamé functions [18,19] allows to find

a particular solution denoted by $\mathfrak{L}(\cdot \mid m; A, B, j)$. Therefore, if small and smooth deformations are assumed, a suitable deformation function Λ is given by

$$\Lambda(\xi) = \mathfrak{L}(q_\eta \sinh b_0 \ \xi \mid 1 + \widetilde{m}; A, B, j) \,. \tag{30}$$

The geometric frustration becomes evident in considering the Euler-Lagrange equation (29a): $\Lambda = 0$ (i.e. the rigid torus section) is not a solution. Now let us turn our attention to the deformation mechanism. As we have seen, the magnetic energy E_{magn} of the configuration (20) is a decreasing function of the eccentric angle b: therefore, for sufficiently small and smooth deformations, the soliton tries to increase the eccentric angle b. From a physical point of view and according to (4), the soliton tends to collapse the torus section ($R \to a^+$ and $r \to 0^+$ as $b \to \infty$). On the other hand, the elastic Hamiltonian (25) tries to maintain the spontaneous shape of the deformable support: the bending rigidity tends to "attract" the eccentric angle b to the spontaneous eccentric angle b_0. The underlying physics makes a good sense: rigidity prevents the support from a collapse … To sum up, any alteration of the eccentric angle b increases the elastic energy whereas any increasing (resp. decreasing) of the eccentric angle b decreases (resp. increases) the magnetic energy —at least for small and smooth deformations. Consequently, the competition between the magnetic energy and the elastic energy induces an augmentation of the eccentric angle b. Further more, according to (11) the soliton energy is essentially localized in the spread zone: the alteration is less important in the spin flip region.

To understand the geometric meaning of the frustration release, let us introduce the *outer radius* $\overline{R} = R + r$ and the *inner radius* $\underline{R} = R - r$. Their geometrical interpretation is transparent. Now, noting that $\overline{R}\,\underline{R} = a^2$, we define the *relative dilatation* λ by

$$\lambda = \frac{\overline{R}}{\overline{R}_0} = \left(\frac{\underline{R}}{\underline{R}_0}\right)^{-1}, \tag{31}$$

where $\overline{R}_0$ and $\underline{R}_0$ are related to the spontaneous shape. The transformation relationships (5) allows to write a readable relation between the relative dilatation λ and the eccentric angle b:

$$\lambda = \frac{\tanh \frac{1}{2} b_0}{\tanh \frac{1}{2} b}. \tag{32}$$

Clearly, the augmentation of the eccentric angle b leads to a global shrinking whereas a local swelling arises where the spins twist, as shown in Fig. 1.

In conclusion, we have shown that the Euler-Lagrange equation for the nonlinear σ-model on a torus section with homogeneous boundary conditions is the sine-Gordon equation: the spread of the topological soliton configurations is tuned by the torus eccentricity. Besides, the model presents a geometrical frustration induced by the torus eccentricity. As announced, the frustration release is shown to lead to a novel geometric effect: the torus section is *globally shrunk* and a swelling appears in the region of the soliton. The full torus problem will be considered in a subsequent paper [18].

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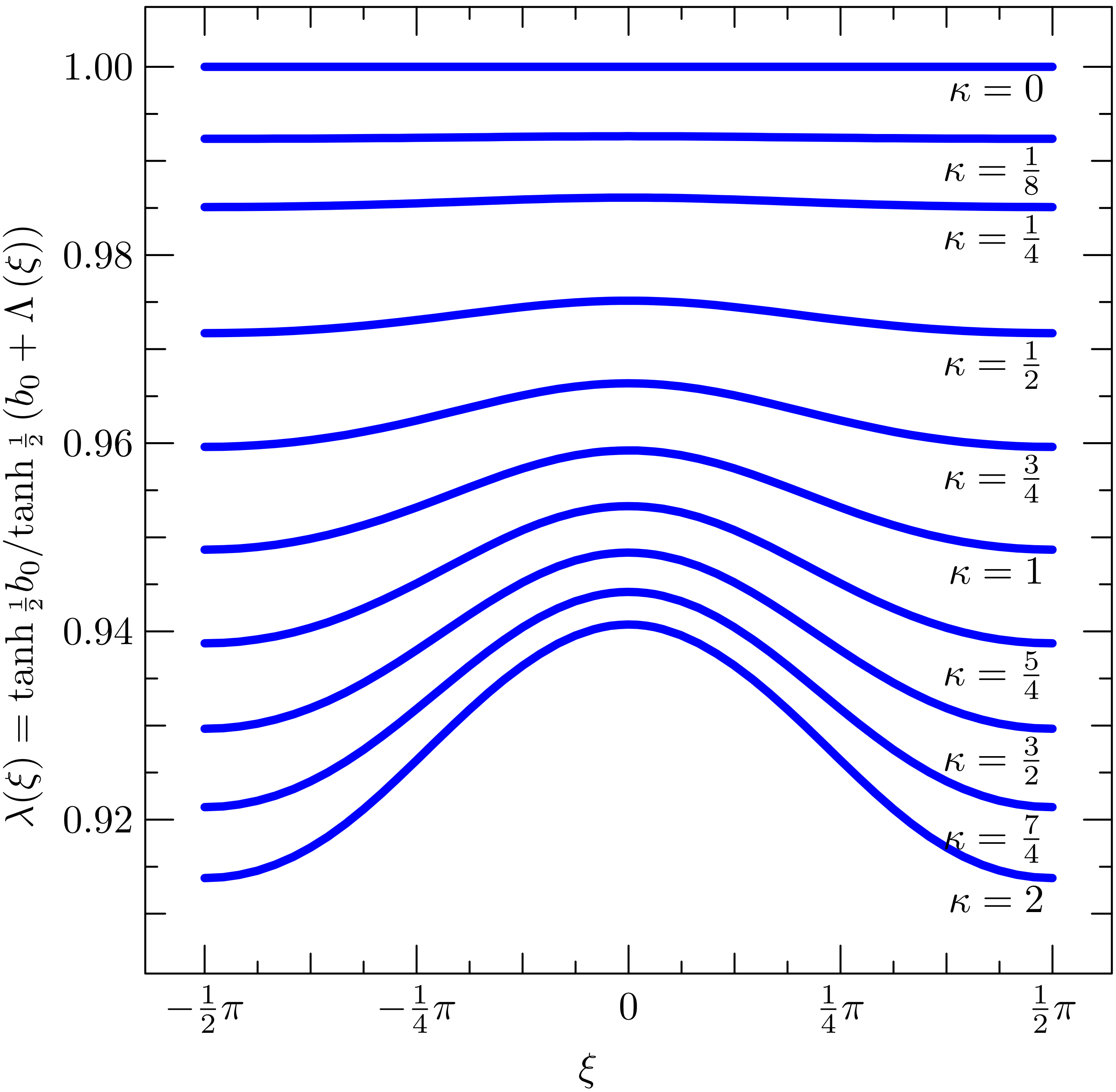

FIG. 1. The relative dilatation $\lambda(\xi)$ as defined in (31) corresponding to the deformation function (30) associated with the π-Clifford torus section ($2\Delta\xi = \pi$ and $\sinh b_0 = 1$) in presence of a π-soliton (20) versus the rotating angle ξ for different values of the relative coupling energy $\kappa \equiv J/k_c$: the case $\kappa = 0$ corresponds to the rigid torus section.